THE COMPLETENESS OF THE IRREDUCIBLE REPRESENTATIONS OF A COMPACT CONTINUOUS GROUP

by

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Abstract. — This is the fundamental paper by Peter and Weyl [4], translated by Alexandre Afgoustidis. I have made very few changes: the references are given at the end of the paper rather than in footnotes, and I made very small changes in vocabulary or notation in order to avoid confusion with now-standard terminology. But overall this is a rather literal translation of the Peter–Weyl paper. My knowledge of German is quite superficial, and I apologize for any mistakes.

1. Basic notions; orthogonality relations

1.1. Groups; volume measurements in the group manifold. — In what follows, we discuss the *representations of a compact topological group by linear transformations*, or *matrices*. In the given group G, we denote by 1_G the unit element. In order to guarantee the existence of an invariant volume form on the group, we shall assume that the infinitesimal notation of Lie is applicable to the group G. We will also assume that the elements infinitesimally adjacent to 1_G form a linear space of dimension r. To any element a of G, we associate a transformation $s \mapsto s'$ from G to itself by means of the formula $s' = sa$. If we call this a (right) *translation*, then the volume form on G is determined by the requirement that it should be invariant under right translations. If s, s' are two infinitesimally close elements (or "points") of G, we understand by the vector from s to s' the infinitesimal element $s's^{-1}$, or its components with respect to a fixed basis of the infinitesimal group. As the *volume* of a paralellepipedic volume element which is spanned by r vectors emerging from s, we understand the absolute value of the determinant of the components of the components of these vectors. For integrations extending over the entire compact group G, we write ds for the volume of such a volume element located at the point s. It is an important fact that ds is invariant under the following transformations $s \mapsto s'$:

1) $s' = as$ (left translation)
2) $s' = s^{-1}$ (inversion).

To see this, one must show that when r infinitesimal elements δs are related to r vectors $\delta s'$ through

$$\delta s' = a \cdot \delta s \cdot a^{-1}, \tag{1}$$

the absolute values of their determinants coincide; or that the transformation A, corresponding in the adjoint group to the element a of G, has a determinant D of absolute value 1. But because of the compactness of the adjoint group, the determinants D^ν that arise from A by iteration must admit, as $\nu \to \infty$, a nonzero cluster value; and therefore we do in fact have $|D| = 1$.

1.2. Representations. — A *representation of dimension* n arises when one attaches to any element s of the group, in continuous fashion, a square matrix $E(s)$ of n lines and columns, in such a way that we always have

$$E(st) = E(s)E(t). \tag{2}$$

By introducing another coordinate system in the representation space, we can transform $E(s)$ into the "equivalent" representation $AE(s)A^{-1}$ (here A is an arbitrary constant matrix with nonzero determinant). The matrix $E(1_G) = C$ must satisfy the relations

$$C \cdot C = C; \quad C \cdot E(s) = E(s) \cdot C = E(s).$$

From the first, it follows that in an appropriate choice of coordinate system, C takes the form

$$C = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{1}_m$ denotes the unit matrix of size m ($m \leq n$). The other two equalities imply that the components of $E(s)$ can be different from 0 only in the subsquare in which C is filled by the identity matrix $\mathbf{1}_m$. Thus one can reduce to the representations for which $E(1_G)$ is the identity matrix; the others representations are equivalent to those which one obtains by adding a corner of zeroes. In particular, in an *irreducible* or *primitive* representation, the matrix $E(1_G)$ must be equal to the identity matrix. (As is well-known, a representation is called reducible when it is possible to locate, inside n-dimensional space, a linear subspace of fewer dimensions which is stable under all transformations $E(s)$.)

1.3. Characters. — The trace $\chi(s)$ of $E(s)$ is called its character. Two equivalent representations have the same character. The character $\chi(s)$ is a class function, that is, we have

$$\chi(asa^{-1}) = \chi(s) \quad \text{or } \chi(st) = \chi(ts)$$

for any two elements a, s or s, t in the group G. This is because the traces of

$$E(st) = E(s)E(t) \quad \text{and} \quad E(ts) = E(t)E(s)$$

agree, on account of the fact that the trace of a product of two matrices does not depend on the order of the factors.

1.4. Orthogonality relations. Unitarizability of representations. — *Every representation* (for which $E(1_G)$ is the identity) *is equivalent to one for which the matrices $E(s)$ are unitary.* We shall use a bar to denote complex conjugation, and a star to denote the transpose operation on matrices. Consider the standard Hermitian form

$$x_1\overline{x_1} + x_2\overline{x_2} + \cdots + x_n\overline{x_n}, \tag{3}$$

apply the transformation $E(s) = (e_{ik}(s))$, and integrate the resulting s-dependent form over the whole group G, using the invariant volume form. One obtains a positive definite Hermitian form H, which is invariant under each of the transformations $E(s)$. By an appropriate change of coordinates, one can bring H to the form (3); then the transformations $E(s)$ become in fact unitary. The form H is

$$= \int_G \sum_{i=1}^{n} |\sum_{k=1}^{n} e_{ik}(s)x_k|^2 ds, \tag{4}$$

and its coefficient matrix is

$$= \int_G E^*(t)\overline{E}(t)dt. \tag{5}$$

Formula (4) shows that H is in fact positive definite when $E(1_G)$ is the identity matrix, and the claimed invariance follows from (5) by the following calculation:

$$E^*(s)H\overline{E}(s) = \int_G E^*(ts)\overline{E}(ts)dt = \int_G E^*(t)\overline{E}(t)dt = H.$$

Once we bring H to standard form, we will also have

$$E^*(s)\overline{E}(s) = \mathbf{1},$$

and since

$$E(s)E(s^{-1}) = E(1_G) = \mathbf{1}, \quad E(s^{-1}) = E(s)^{-1},$$

this can be alternatively written as

$$E(s^{-1}) = \overline{E}^*(s).$$

For the characters, it follows that

$$\chi(s^{-1}) = \overline{\chi}(s).$$

In a similar way, I. Schur has shown that for two non-equivalent irreducible representations

$$E(s) = (e_{ik}(s)), \quad E'(s) = (e'_{ik}(s)),$$

the following relations hold ([**5, 6**], see also [**7**]):

$$\int_G e_{ik}(s)e'_{\kappa\iota}(s^{-1})ds = 0. \tag{6}$$

On the other hand, for an individual irreducible representation, the following relations follow from the fact that any constant matrix commuting with all $E(s)$ must be a multiple of the identity:

$$\int_G e_{ik}(s)e_{\iota\kappa}(s^{-1})ds = \begin{cases} \frac{V}{n} & \text{for } i = \iota,\ k = \kappa, \\ 0 & \text{in all other cases.} \end{cases} \tag{7}$$

Here $V = \int_G ds$ denotes the total volume of G. If we assume, as we may, that all representation matrices are unitary, then the left sides of (6), (7) become

$$\int_G e_{ik}(s)\overline{e}'_{\iota\kappa}(s)ds \qquad \text{and} \qquad \int_G e_{ik}(s)\overline{e}_{\iota\kappa}(s)ds.$$

Therefore the components of the various inequivalent representations $E(s)$ form an orthonormal system of functions on the manifold G. It immediately follows that they are linearly independent of one another[(1)]. *In what follows, we will prove the completeness of this orthogonal system.* For the primitive characters, the orthogonality relations mentioned above imply

$$\int_G \chi(s)\overline{\chi}'(s)ds = 0, \quad \int_G \chi(s)\overline{\chi}'(s)ds = V. \tag{8}$$

The first holds when χ, χ' are the characters of two inequivalent irreducible representations; it follows readily that such characters must be different from one another.

[(1)]For finite groups, this independence was proved by G. Frobenius and I. Schur [**3**], following a method which Burnside had used towards the same goal for the components of a single irreducible representation.

For comparison with what follows, it will be useful to indicate a proof of the fact that every matrix A commuting with all $E(s)$ must be a multiple of the identity matrix, by a somewhat different argument than that given by I. Schur. We want to show how, given a matrix A commuting with all $E(s)$ and not equal to a multiple of the identity, we can bring about a decomposition of the representation into separate components. Since

$$E(s)A = AE(s) \quad \text{or} \quad E(s)A\overline{E}^*(s) = A,$$

the same properties hold with $\overline{A}^*$ instead of A. Among the Hermitian matrices

$$A + \overline{A}^*, \qquad i(A - \overline{A}^*),$$

which can be linearly combined to reconstruct A, there must be at least one which is nonscalar. We obtain in this way a Hermitian form invariant under all $E(s)$, which we will denote again by A. Then we can use a unitary transformation of the coordinate system to bring A into the form

$$A = \alpha_1 x_1 \overline{x_1} + \alpha_2 x_2 \overline{x_2} + \cdots + \alpha_n x_n \overline{x_n},$$

where it is impossible for all coefficients α_i to be equal to one another. The matrices $E(s)$, which commute with A, will then break down in block-diagonal form, just as the quantities α_i divide (under an appropriate arrangement) into parts where the values are equal to one another.

2. Bessel's inequality. Approach to the problem.

Given a continuous function x on the manifold G, we can build the matrix and the number which play for our orthogonal system the role of Fourier coefficients:

$$A(x) = \int_G x(s)E^*(s^{-1})ds = \int_G x(s)\overline{E}(s)ds,$$

and its trace

$$\alpha(x) = \mathrm{Tr}(A(x)) = \int_G x(s)\chi(s^{-1})dg = \int_G x(s)\overline{\chi}(s)ds.$$

It follows from the above-established orthogonality relations, applied to the components $\alpha_{ij}(x)$ of the matrix $A(x)$, the Bessel inequalities

$$n\sum_{i,k}|\alpha_{ik}(x)|^2 + \ldots \leq V\int_G |x(s)|^2 ds, \tag{9}$$

$$|\alpha(x)|^2 + \ldots \leq V\int_G |x(s)|^2 ds. \tag{10}$$

In the sum on the left, $+\ldots$ denotes a number of analogous terms that arise from inequivalent irreducible representations. The assertion of completeness means that if all representations are included on the left, then the first relation always becomes an equality, and the second becomes an equality when x is a class function.

We shall view each function $x(s)$ on the group manifold as a *group number. Addition* has the usual meaning, but *multiplication* is defined by

$$(xy)(s) = \int_G x(sr^{-1})y(r)dr.$$

We can also write it more symmetrically as

$$(xy)(st^{-1}) = \int_G x(sr^{-1})y(rt^{-1})dr$$

by substituting rt^{-1} for r in the last integral.

Note that the set of *kernels of the form* $k(s,t) = x(st^{-1})$ is closed under the operations of addition and composition. The trace of such a kernel, which we shall also call the trace of the group number x, is

$$S(x) = V \cdot x(1);$$

therefore

$$S(xy) = V \cdot \int_G x(s^{-1})y(s)ds = V \cdot \int_G x(s)y(s^{-1}) = S(yx).$$

As (Hermitian) conjugate of the group number $x(s)$, we set

$$\widetilde{x}(s) = \overline{x}(s^{-1});$$

this agrees with the well-known definition

$$\widetilde{k}(s,t) = \overline{k}(s,t)$$

for the corresponding kernels. A group number x (or the corresponding kernel) is called Hermitian when the equality $x = \widetilde{x}$ holds.

Now the matrix

$$A(\widetilde{x}) = \int_G \overline{x}(s^{-1})\overline{E}(s)ds = \int_G \overline{x}(s)\overline{E}(s^{-1})ds$$

is the conjugate-transpose of

$$\int_G x(s)E^*(s^{-1})ds = A(x),$$

and therefore we also have

$$\alpha(\widetilde{x}) = \overline{\alpha}(x).$$

We can then write the Bessel inequalities in the form:

$$n\mathrm{Tr}(A(x)A(\widetilde{x})) + \ldots \le S(x\widetilde{x}); \tag{11}$$

$$\alpha(x)\alpha(\widetilde{x}) + \ldots \le S(x\widetilde{x}). \tag{12}$$

The defining property of a representation (2) corresponds to the multiplication rule

$$A(xy) = A(x)A(y).$$

Indeed, the product on the right equals $\int_G \int_G x(s)y(t)\overline{E}(st)dsdt$, and if one replaces s (for fixed t) by st^{-1}, one obtains

$$A(x)A(y) = \int_G \int_G x(st^{-1})y(t)\overline{E}(s)dsdt$$

and by swapping the order of the integration this gives $A(xy)$.

Furthermore, we have the important relations

$$E(s)A^*(x) = \int_G x(t)E(st^{-1})dt = \int_G x(ts)E(t^{-1})dt, \tag{13}$$

$$A^*(x)E(s) = \int_G x(t)E(t^{-1}s)dt = \int_G x(st)E(t^{-1})dt. \tag{14}$$

If x is a class function, then $A^*(x)$ must commute with all elements $E(s)$, and therefore it must be a multiple of the identity matrix of size n:

$$A(x) = \frac{\alpha(x)}{n} \cdot \mathbf{1}.$$

As a result, the special completeness relation (12), which refers to class functions, follows from the general completeness relation (11).

We shall obtain the completeness relation from the theory of eigenvalues and eigenfunctions for kernels of the particular form $x(st^{-1})$.

Let $z = x\,\widetilde{x}$ be the Hermitian kernel resulting from composition with the conjugate; then (14) says

$$\int_G z(st^{-1})E(t)dt = \Gamma^* E(s) \tag{15}$$

where

$$\Gamma = A(z) = A(x)\overline{A}^*(x)$$

is a Hermitian matrix; the corresponding Hermitian form can have no negative eigenvalue. By applying a suitable unitary transformation to the representation $E(s)$, we can turn Γ into a diagonal matrix, whose (real, nonnegative) coefficients we will denote by $\gamma_1, \gamma_2, \ldots, \gamma_n$. We then recognize from (15) that the functions

$$e_{i1}(s), e_{i2}(s), \ldots, e_{in}(s)$$

are eigenfunctions of the kernel $z(st^{-1})$ for the eigenvalue γ_i.

The theory of integral equations, if applied to this kernel, will give us those irreducible representations of the group for which the Fourier coefficient $A(x)$ does not vanish; and these representations will arise constructively from a diagonalization procedure which depends on x.

The announced Completeness relation then reduces to the theorem, well-known in the theory of integral equations, that the trace of the kernel $z = x\,\widetilde{x}$ is exactly the sum of its eigenvalues. By constructively generating the irreducible representations, we will derive all the results discussed in §1 anew, but beyond that we will gain completeness(2). In the simplest case of the commutative one-parameter group of rotations of the circle, the method has already been explained in the proceedings of the Berlin meeting (1926), p. 211. It leads to the well-known Parseval identity in the theory of Fourier series; as shown in *loc. cit.* and in [**8**], it can also be used very nicely to obtain the more general Bohr completeness relation in the theory of almost periodic functions.

3. Construction of the highest representation to which a group number belongs

For constructing eigenvalues and eigenfunctions, we use (with minor modifications, already given in [**8**] just cited) the method developed by E. Schmidt in his 1905 Dissertation. The main task is, beginning with a function $x(s)$ on G which is not identically zero, to construct a representation $E(s)$ for which $A(x) \neq 0$. Since every representation of a compact group is completely reducible, we can produce from that an *irreducible* representation with the same property.

We can iterate the Hermitian kernel $z = x\widetilde{x}$, and consider z, z^2, z^3, $\ldots$. The traces σ_ν $(\nu = 1, 2, 3, \ldots)$ of the iterates are positive numbers, and the quotients $\sigma_\nu/\sigma_{\nu-1} = \gamma_\nu$ increase to a positive value γ, while $\frac{z^\nu(s)}{\gamma^\nu}$ converges uniformly on G to a limit $e(s)$. The group number $e(s)$ is Hermitian, its trace is ≥ 1. The number γ is the largest eigenvalue of $z(st^{-1})$, and $e(st^{-1})$ belongs to the corresponding eigenspace for $z(st^{-1})$. We have the relations

$$z\,e = e\,z = \gamma e, \quad e\,e = e.$$

(2) The result and the strategy of the proof have already been announced in [**7**, p. 390].

Because of the last equality, we can decompose $e(st^{-1})$ in the following way:

$$e(st^{-1}) = \varphi_1(s)\overline{\varphi_1}(t) + \varphi_2(s)\overline{\varphi_2}(t) + \cdots + \varphi_n(s)\overline{\varphi_n}(t)$$

where the $\varphi_i(s)$ form an orthonormal system

$$\int_G \varphi_i(s)\overline{\varphi_k}(s)ds = \delta_{ik} = \begin{cases} 1 & (i = k) \\ 0 & (i \neq k) \end{cases}$$

of eigenfunctions of z for the eigenvalue γ.

Since for fixed t, the map $s \mapsto \varphi_i(st^{-1})$ is also an eigenfunction of z for the eigenvalue γ, we must have equalities

$$\varphi_i(st^{-1}) = \sum_{k=1}^{n} \overline{e_{ik}}(t)\varphi_k(s). \tag{16}$$

These equalities follow from

$$\varphi_i(s) = \int_G e(sr^{-1})\varphi_i(r)dr,$$

if one substitutes st^{-1} for s and insert the expression

$$e(st^{-1}r^{-1}) = \sum_{k=1}^{n} \varphi_k(s)\overline{\varphi_k}(rt),$$

leading to the equality

$$\overline{e_{ik}}(t) = \int_G \varphi_i(r)\overline{\varphi_k}(rt)dr.$$

Thus, by the change of variable $s \mapsto s' = st^{-1}$, the functions $\phi_i(s)$ transform according to the linear substitution

$$\overline{E}(t) = (\overline{e}_{ik}(t))_{i,k=1\ldots n}.$$

It follows immediately that

$$\overline{E}(t')\overline{E}(t) = \overline{E}(t't),$$

in other words, $E(s)$ *is a representation of* G of dimension n. The matrix $E(1_G)$ is the identity. Since the $\varphi_i(s)$ are an orthonormal system of functions

Since the $\varphi_i(s)$ form an orthonormal system of functions and since this property is obviously not affected by the substitution $s \leftarrow st^{-1}$, the linear transformation is unitary:

$$\overline{E}^*(t) = E(t^{-1}), \quad \widetilde{e}_{ik}(t) = e_{ki}(t).$$

Finally, we have to check that for this representation the matrix

$$A(z) = \Gamma = (\gamma_{ik})$$

is nonzero. The quickest way to do this is probably as follows. From (16) it follows that

$$\int_G \varphi_i(st^{-1})z(t)dt = \sum_k \gamma_{ik}\varphi_k(s),$$

and so

$$\int_G \varphi_i(t^{-1})z(t)dt = \sum_k \gamma_{ik}\varphi_k(1_G).$$

In addition

$$\int_G z(st^{-1})e(t)dt = \gamma e(s),$$

therefore

$$\gamma e(1_G) = \frac{\gamma n}{V} = \int_G z(t^{-1})e(t)dt = \sum_{i=1}^{n} \overline{\varphi_i}(1_G) \int_G z(t^{-1})\varphi_i(t)dt = \sum_{i,k} \gamma_{ik}\overline{\varphi_i}(1_G)\varphi_k(1_G).$$

The Hermtian form with coeffients γ_{ik} takes the nonzero value $\frac{\gamma n}{V}$ at the argument $\varphi_i(1_G)$, and therefore it cannot vanish identically.

4. Decomposition of the obtained representation

It will next be good to clarify between the representation $E(s)$ and the eigenfunctions $\varphi_i(s)$ in more detail. After that, we shall determine the decomposition of $E(s)$ into irreducible components in as simple a way as possible. Finally, by repeating the above procedure, we will determine all representations attached to the kernel $x(st^{-1})$, thereby completing the proof of the Completeness theorem.

Concerning the first point, Equation 14 gives

$$zE(s) = \int_G z(st^{-1})E(t)dt = \Gamma^* E(s).$$

Applying a suitable unitary transformation, we can turn Γ into a diagonal matrix, with eigenvalues $\gamma_1, \dots, \gamma_n$, $\gamma_i \geq 0$. Then

$$e_{i1}(s), e_{i2}(s), \dots, e_{in}(s)$$

are eigenfunctions of the kernel $z(st^{-1})$ for the eigenvalue γ_i. Now if we take (16), substitute t^{-1} for t and $e_{ki}(t)$ for $\overline{e_{ik}}(t^{-1})$, and set $s = 1_G$, we see that the eigenfunction

$$\varphi_i(t) = \sum_k e_{ki}(t)\varphi_k(1_G) \tag{17}$$

is a linear combination of the columns of $E(t)$. Therefore the eigenvalue γ must be one of the numbers $\gamma_1, \dots, \gamma_n$. [Note that $\varphi_k(1_G)e_{ki}(t)$ must be zero for arbitrary i, k such that $\gamma_k \neq \gamma$; taking $i = k$ and $t = 1_G$, one concludes that $\varphi_k(1_G) = 0$ for $\gamma_k \neq \gamma$.]

Concerning the second point, in order to carry out the reduction, we look for all "characteristic units", that is, those functions for which there is a decomposition

$$f(st^{-1}) = \sum_{i,k} \lambda_{ik}\varphi_i(s)\overline{\varphi_k}(t) \tag{18}$$

with constant coefficients λ_{ik}. According to (18), such a function $f(s)$ can be expressed as a linear combination of $\varphi_1(s), \dots, \varphi_n(s)$; furthermore we must have

$$e\,f = f\,e = f.$$

Therefore $f(s)$ must be a linear combination of

$$(\varphi_1\, e)(s), (\varphi_2\, e)(s), \dots, (\varphi_n\, e)(s). \tag{19}$$

On the other hand, these products are actually characteristic units, because $\varphi_i(sr^{-1})$ can be expressed as a linear combination of the functions $\varphi_j(s)$, while $e(rt^{-1})$ can be expressed as a linear combination of the functions $\overline{\varphi_i}(t)$.

We now check whether the functions (19) can be constant multiples of $e(s)$. If this is the case, then we claim that the representation $E(s)$ is irreducible. Otherwise, (19) provides a characteristic unit $f(s)$ that is not a multiple of $e(s)$. Since the conjugate $\widetilde{f}(s)$ is of the same nature, satisfying

$$\widetilde{f}(st^{-1}) = \sum_{i,k} \overline{\lambda_{i,k}} \varphi_i(s)\overline{\varphi_k}(t),$$

we can produce, by considering

$$f + \widetilde{f} \quad \text{or} \quad i(f - \widetilde{f}),$$

a characteristic unit which is not a multiple of $e(s)$ and is in addition Hermitian. If we now denote the latter by f, then the coefficients of (18) must satisfy $\overline{\lambda_{ik}} = \lambda_{ki}$. By a suitable unitary transformation of the system of eigenfunctions $\varphi_i(s)$ (which was in any case determined only up to such a transformation), we can turn the Hermitian matrix $(\lambda_{ik})_{i,k}$ into a diagonal matrix, and assume $\lambda_{ik} = 0$ for $i \neq k$. If certain of the numbers $\lambda_{ii} = \lambda_i$ satisfy $\lambda_1, \dots \lambda_g = \lambda'$, while the others are $\neq \lambda'$, then we can produce from f, using a linear combination of f and its iterates, a new characteristic unit

$$e'(st^{-1}) = \sum_{i=1}^{q} \varphi_i(s)\overline{\varphi_i}(t).$$

In fact, if $\lambda', \lambda'', \dots$ are the h different values among the n numbers λ_i, it is enough to construct a polynomial of degree $h-1$ in the variable μ

$$\beta_0 + \beta_1\mu + \dots + \beta_{h-1}\mu^{h-1}$$

which takes the value 1 at $\mu = \lambda'$ and vanishes at $\mu = \lambda'', \dots, \lambda^{(h)}$; then for the desired unit, one can take

$$e'(s) = \beta_0\, e(s) + \beta_1\, f(s) + \beta_2\, (f\, f)(s) + \dots \beta_{h-1}\, f^{h-1}(s).$$

From the definition of the matrix $E(t)$, it follows that $E(t)$ splits into block-diagonal form, with h blocks corresponding to the division of the sequence $\lambda_1, \dots, \lambda_n$ in groups of identical numbers. The same is true of the matrix $\Gamma = A(z)$.

For each of the functions $e'(s), e''(s), \dots$ obtained in this way, one can apply the same procedure as just applied to $e(s)$; eventually, after at most n steps in which the system of eigenfunctions $\varphi_i(s)$ gets normalized anew, we can obtain a division into sections

$$(\varphi_1(s), \dots, \varphi_g(s)), (\dots), \dots,$$

with the property that for each part the sum

$$e'(st^{-1}) = \sum_{i=1}^{g} \varphi_i(s)\overline{\varphi_i}(t)$$

is a function of st^{-1} alone, but is a linear combination

$$\sum_{i,k=1}^{g} \lambda_{ik}\varphi_i(s)\overline{\varphi_k}(t)$$

only if it is a constant multiple of $e'(st^{-1})$.

The matrices $E(t)$ and Γ decompose accordingly. By a final normalization, Γ can be brought into the form of a diagonal matrix. Among the eigenvalues $\gamma_1, \dots, \gamma_g$ belonging to a given section, we can always represent γ according to Eq. (17). We still have to prove that the submatrices obtained in this way produce *irreducible* representations.

We change the notation and focus on one of the submatrices constructed above, writing n instead of g. Consider the quantities

$$\int_G \varphi_i(sr^{-1})\overline{\varphi_k}(tr^{-1})dr = \psi(st^{-1}) = \sum_{p,q=1}^{n} \lambda_{pq}\varphi_p(s)\overline{\varphi_q}(t),$$

$$\lambda_{pq} = \int_G \overline{e_{ip}}(r)e_{kq}dr.$$

According to the hypothesis, the matrix $(\lambda_{p,q})_{p,q}$ must be a multiple of the identity:

$$\int_G \overline{e_{ip}}(r)e_{kq}(r)dr = \varrho_{ik}\delta_{pq}.$$

Setting $p = q$ and summing over p, the Orthogonality relations

$$\int_G \sum_p \overline{e_{ip}}(r)e_{kp}(r) = \delta_{ik}$$

imply the formula

$$V\delta_{ik} = n\varrho_{ik}$$

therefore

$$\int_G \overline{e_{ip}}(r)e_{kq}(r)dr = \begin{cases} V/n & \text{for } i = k,\ p = q, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

We have therefore given a new proof of the orthogonality relations (7) between the matrix elements of the subrepresentation under inspection. Because of these orthogonality relations, the components $e_{ik}(s)$ must be linearly independent, and thus the representation must be irreducible. If we use an arbitrary matrix $A = (\alpha_{ik})_{i,k}$ to construct the function

$$y(s) = \frac{n}{V}\sum_{i,k} \alpha_{ik}e_{ik}(s),$$

then it follows from (20) that $A(y) = A$, and (14) yields

$$ye_{qk} = \sum_i \alpha_{iq}e_{ik}.$$

Thus the relations (20) expand to give

$$e_{ip}e_{qk} = \begin{cases} 0 & (p \neq q), \\ \frac{V}{n}e_{ik} & (p = q). \end{cases} \tag{21}$$

The discussion here is obviously closely related to the recapitulations in §1, which go back to I. Schur.

In order to make this completely clear, let us prove that for an *arbitrary* collection of n eigenfunctions $\varphi_i(s)$ for the eigenvalue γ of the kernel $z(st^{-1})$, and for the corresponding representation $E(s)$ of dimension n (which is not necessarily irreducible), the following theorem holds: the function

$$\sum_{i,k=1}^{n} \lambda_{ik}\varphi_i(s)\overline{\varphi_k}(t) \tag{22}$$

is of the form $f(st^{-1})$ if and only if the matrix $\Lambda = (\lambda_{ik})_{i,k}$ commutes with all matrices $E(s)$.

Indeed, the sum (22), which we will denote by $f(s,t)$, has the desired form if and only if

$$f(sr, tr) = f(s,t) \tag{23}$$

for all r. Now, apply

$$\varphi_i(sr) = \sum_p e_{pi}(r)\varphi_p(s), \quad \overline{\varphi_k}(tr) = \sum_q \overline{e_{qk}}(r)\overline{\varphi_q}(s).$$

Condition (23) becomes

$$\sum_{i,k} \lambda_{ik} e_{pi}(r)\overline{e_{qk}}(r) = \lambda_{pq},$$

in other words

$$E(r)\Lambda\overline{E}^*(r) = \Lambda.$$

5. Integration; proof of the completeness theorem

Given a group number x which is not identically zero, we have constructed a positive number γ and an irreducible representation $E(s) = (E_{ik}(s))_{i,k}$ of our group G, in such a way that $A(x)\overline{A}^*(x) = \Gamma$ is a diagonal matrix whose elements $\gamma_1, \dots, \gamma_n$ satisfy $0 \leq \gamma_i \leq \gamma$ and such that at least one of the γ_i is equal to γ. Thus the trace of Γ is $\geq \gamma$.

Let us now substract from $x(s)$ the combination which corresponds to the determined eigenfunctions:

$$x(s) = \frac{n}{V}\sum_{i,k} \alpha_{ik} e_{ik}(s) + x'(s) \quad \alpha_{ik} = \alpha_{ik}(x).$$

Taking into account (21) and the fact that $\widetilde{e}_{ik} = e_{ki}$, we find immediately that for $z' = x'\,\widetilde{x}'$, we have

$$z(s) = \frac{n}{V}\sum_i \gamma_i e_{ii}(s) + z'(s),$$

and by iteration

$$z^\nu(s) = \frac{n}{V}\sum_i \gamma_i^\nu e_{ii}(s) + (z')^\nu(s).$$

The trace σ'_ν of $(z')^\nu$ becomes

$$\sigma'_\nu = \sigma_\nu - n\left(\gamma_1^\nu + \dots + \gamma_n^\nu\right).$$

If x' is not identically zero, then $\sigma'_\nu/\sigma'_{\nu-1}$ converges as $\nu \to \infty$ to some limit γ'. We cannot have $\gamma' > \gamma$, because $\frac{\sigma'_\nu}{(\gamma')^\nu} \leq \frac{\sigma_\nu}{(\gamma')^\nu}$ converges, like $\frac{\sigma_\nu}{\gamma^\nu}$, to an integer $n' \geq 1$. Using the part of the kernel $z'(st^{-1})$ corresponding to the eigenvalue γ', one obtains a function

$$e'(st^{-1}) = \varphi'_1(s)\overline{\varphi_1}'(t) + \dots \qquad (n' \text{ terms}).$$

From $z'e_{ik} = 0$ it follows that $e'e_{ik} = 0$, that is,

$$\int_G \overline{\varphi_p}'(t)e_{ik}(t)dt = 0 \quad \text{or} \quad \int_G \varphi_p{}'(t)\overline{e_{ik}}(t)dt = 0.$$

By (14), we even have

$$\int_G \varphi'_p(st^{-1})e_{ik}(t)dt = 0,$$

and if we arrange matters so that

$$\varphi'_p(st^{-1}) = \sum_q e'_{pq}(t)\varphi'_q(s),$$

we get

$$\int_G \overline{e_{p,q}}'(t)e_{ik}(t)dt = 0.$$

These relations also apply to each of the irreducible components of the representation $E'(s) = (e'_{pq})_{p,q}$. Any such irreducible constituent, which we will henceforth denote $E'(s)$, must therefore be non-equivalent to $E(s)$, and we have again obtained the orthogonality relations (6), which moreover show that the associated Fourier coefficient is

$$A'(z') = A(z').$$

If we repeat this process p times, without first reaching an end by exhausting the function $x(s)$, then there arises from $x(s)$ the function

$$x^{(p)}(s) = x(s) - \left(\frac{n}{V} \sum_{i,k} \alpha_{ik} e_{ik}(s) + \dots \right) \qquad (p \text{ terms})$$

with associated kernel

$$z^{(p)}(s) = z(s) - \left(\frac{n}{V} \sum_{i} \gamma_i e_{ii}(s) + \dots \right).$$

Next, we have

$$n \sum_i \gamma_i + \dots \leq V \cdot z(1_G) :$$

therefore, if we assume for simplicity that $z(1_G) = 1$, we have *a fortiori*

$$\gamma + \gamma' + \dots \leq V.$$

Since $\gamma \leq \gamma' \dots$, we have $\gamma^{(p-1)} \leq V/p$, and by construction

$$\sigma_2^{(p)} \leq \gamma^{(p-1)} \sigma_1^{(p)},$$

i.e.

$$\int_G |z^{(p)}(s)|^2 ds \leq \gamma^{(p-1)} \leq \frac{V}{p}. \tag{24}$$

If the process goes on indefinitely, without ever stopping, then $z^{(p)}(s)$ converges uniformly to 0 as $p \to \infty$. Indeed, for any $\varepsilon > 0$, we must have

$$\int_G |x(ts) - x(s)|^2 ds \leq \varepsilon^2$$

as soon as t is in a certain neighborhood G_ε of the identity with volume V_ε. This follows from $z = x\,\widetilde{x}$, $z^{(p)} = x^{(p)}\,\widetilde{x}$ and from an application the Schwarz inequality, using the fact that

$$|z(st^{-1}) - z(s)| \leq \varepsilon, \quad |z^{(p)}(st^{-1}) - z^{(p)}(s)| \leq \varepsilon$$

for all t in the neighborhood G_ε, and for all s. All the functions $z^{(p)}(s)$ are therefore "similarly" uniformly continuous. If at a point we have $|z^{(p)}(s)| \geq \varepsilon$, then the function $z^{(p)}(s)$ remains $\geq \varepsilon$ in absolute value in a whole neighborhood of this point of volume V_ε, and the integral on the left of (24) is at least $\varepsilon^2 V_\varepsilon$, thus $p \leq \frac{V}{\varepsilon^2 V_\varepsilon}$. We deduce that

$$|z^{(p)}(s)| \leq 2\varepsilon, \quad \text{as soon as} \quad p > \frac{V}{\varepsilon^2 V_\varepsilon}.$$

This not only proves our claim of uniform convergence, but also gives an *explicit estimate of the remainder*. In particular, we obtain for $s = 1_G$:

$$n \mathrm{Tr}(A(x) A(\widetilde{x})) + \dots = S(x\,\widetilde{x}). \tag{25}$$

The sum on the left refers to the inequivalent irreducible representations "attached to $x(s)$", namely those provided by our construction method starting from the function $x(s)$. Because of the Bessel inequality, however, (25) holds *a fortiori* if the sum is extended over all inequivalent irreducible representations; and at the same time it turns out that the representations attached to $x(s)$ are those for which the Fourier coefficient $A(x)$ is nonzero.

Fundamental theorem. — *If one attaches to every irreducible representation*

$$E(s) = (e_{ik}(s))_{i,k=1,\dots,n}$$

and to its character $\chi(s)$*, the Fourier coefficients*

$$\alpha_{ik} = \int_G x(s)\overline{e_{ik}}(s)df, \qquad \alpha = \int_G x(s)\overline{\chi}(s)ds,$$

then the equality

$$n\sum_{i,k}|\alpha_{ik}|^2 + \cdots = V \cdot \int_G |x(s)|^2 ds$$

holds for every continuous function $x(s)$*, and we have*

$$\alpha^2 + \cdots = V \cdot \int_G |x(s)|^2 ds$$

when $x(s)$ *is a class function. The left sum is over all inequivalent irreducible representations.*

The main difference between our proof and the well-known argument which one can apply towards the same goal in the theory of finite groups[(3)] is that here, we cannot use a group number "**1**" with the property

$$\mathbf{1}\,x = x\,\mathbf{1} = x.$$

Therefore, the proof had to be rearranged to work, instead of **1**, with an arbitrary group number $x(s)$. But the theory of integral equations then provides the completeness relation for $x(s)$ directly. If one wants to constructively generate *all* inequivalent irreducible representations in this way, one must use a sequence of functions $\mathbf{1}_\nu(s)$ which converge to that unrealizable **1**. Thus take for $\mathbf{1}_\nu(s)$ a nonnegative function, vanishing outside of a neighborhood U_ν of the unit element which shrinks as $n \to \infty$ to the point 1_G. Let the integral of $\mathbf{1}_\nu$ be $= 1$. Then in fact every irreducible representation must enter the picture; indeed, for a given such representation $E(s)$, the corresponding

$$A(\mathbf{1}_\nu) = \int_G \mathbf{1}_\nu(s)\overline{E}(s)ds$$

clearly converges, as $\nu \to \infty$, to $\overline{E}(1_G)$, *i.e.* to the identity matrix. Therefore $A(\mathbf{1}_\nu)$ is definitely nonzero for ν large enough, and then $E(s)$ cannot vanish on $\mathbf{1}_\nu$.

6. Expansion theorem; approximation theorem; applications

Let x, y be group numbers, and consider the product

$$u(s) = \int_G x(st^{-1})y(t)dt.$$

Because

$$|x^{(p)}y|^2 \le \int_G |x^{(p)(t)}|^2 dt \cdot \int_G |y(t)|^2 dt$$

(3) Here it is appropriate to name Frobenius and his fundamental work [2], and Burnside, who brought together his methods and result in the book [1].

converges uniformly to 0, we have the following *uniformly convergent Fourier expnsion for $u(s)$:*

$$\begin{aligned} V \cdot u(s) &= n \operatorname{Tr}\left(\int_G E(st^{-1}) y(t) dt \cdot A^*(x) \right) + \dots \\ &= n \operatorname{Tr}\left(E(s) A^*(y) A^*(x) \right) + \dots \\ &= n \sum_{i,k} \alpha_{i,k}(u) e_{ik}(s) + \dots \end{aligned}$$

The sum needs only the irreducible representations attached to $x(s)$. This is the *expansion theorem.* It contains the following form of the completeness relation:

$$S(x\, y) = n \operatorname{Tr}(A(x)A(y)) + \dots$$

If we take for y the functions denoted by $\mathbf{1}_\nu$ above, we obtain a sequence $u_\nu(s)$ of functions which converge uniformly to $x(s)$. From the expansion theorem, we deduce the *approximation theorem*, which says that *every continuous function $x(s)$ on G can be uniformly approached by a finite sum of the form*

$$\sum_{i,k} \beta_{ik} e_{ik}(s) + \dots$$

in whose components enter only irreducible representations for which the Fourier coefficient $A(x)$ does not vanish.

Now if x and y are class functions, then $u = x\, y$ is also a class function, and it admits a *uniformly convergent Fourier expansion in terms of characters*

$$V \cdot u(s) = \alpha(u) \cdot \chi(s) + \dots .$$

In order to deduce from this expansion theorem an approximation theorem, we need to use a sequence of *class functions* $\mathbf{1}^*_\nu(s)$ with properties analogous to those of the functions $\mathbf{1}_\nu(s)$ used above. The easiest way is to define

$$\mathbf{1}^*_\nu(s) = \frac{1}{V} \int_G \mathbf{1}_\nu(t^{-1} s t) dt.$$

As s runs through the neighborhood U_ν and t runs through the whole group, the element $t^{-1}st$ runs through a neighborhood U^*_ν containing U_ν, which, just as U_ν, shrinks to the identity element as $\nu \to \infty$. Thus we obtain the following theorem: *every continuous class function $x(s)$ can be approximated arbitrary close by finite linear combinations of those primitive characters $\chi(s)$ for which $\int_G x(s)\overline{\chi}(s) ds \neq 0$.*

The most important application of the completeness relation, however, can be expressed by the following two statements:

I. *If s_0, t_0 are elements of G such that $E(s_0) = E(t_0)$ for all irreducible representations E, then s_0 and t_0 coincide.*
II. *If $\chi(s_0) = \chi(t_0)$ for all primitive characters χ, then s_0 and t_0 must belong to the same conjugacy class.*

Statement I. follows already from the result of §3. Indeed, set

$$s_0\, t_0^{-1} = a, \quad s_0 = a\, t_0,$$

then we have $E(a) = \mathbf{1}$. Thus we have $E(sa^{-1}) = E(s)$ for all s and therefore

$$\int_G \overline{E}(sa^{-1}) x(s) ds = \int_G \overline{E}(s) x(sa) ds = \int_G \overline{E}(s) x(s) ds.$$

If $x(sa)-x(s)$ were not identically zero, there would exist an irreducible representation satisfying

$$\int_G (x(sa) - x(s))\overline{E}(s)ds \neq 0.$$

As a result, we must have $x(sa) = x(s)$ for all s, in particular, $x(a) = x(1_G)$, for every continuous function x on G. So $a = 1_G$.

In order to prove II., we must take a detour using the approximation theorem. If $x(s)$ is any class function, and if we approximate it at error ε by a linear combination of characters, then the assumption that $\chi(s_0) = \chi(t_0)$ for all χ implies

$$|x(s_0) - x(t_0)| \leq 2\varepsilon,$$

and since we can make ε arbitrarily small, we must have

$$x(s_0) = x(t_0). \tag{26}$$

Let us choose around x_0 a small neighborhood U; using a continuous function $y(s)$ which is positive in U and vanishes in $G \setminus U$, let us define

$$x(s) = \int_G y(r^{-1}sr)dr,$$

then $x(s_0) \neq 0$. If t_0 were conjugate to none of the elements of U, we could deduce from (26) that $x(t_0) = 0$.

Thus t_0 must be conjugate to elements lying in any neighborhood of s_0; because of the compactness of the group, this implies that t_0 is conjugate to s_0 itself.

Our investigations are particularly important for *semisimple groups*. It is true that those are not always compact, but by the "unitary trick[(4)]", one can associate to any such group G a compact and simply connected group G_u, which provides all the representations and characters of G.[(5)] Bohr's theory of almost periodic functions is the first example of theory of characters for a truly noncompact group, namely the one-parameter abelian group of translations of a line. For such more far-reaching problems, our method also shows its virtues, as testified by the above-cited work in Math. Annalen [**8**]. We hope to come back to this in a future work.

References

[1] W. Burnside – "Theory of groups of finite order. 2nd edition.", Cambridge: University Press. xxiv, 512 S. (1911)., 1911.

[2] G. Frobenius – "Ueber die Primfactoren der Gruppendeterminante.", *Berl. Ber.* **1896** (1896), p. 1343–1382 (German).

[3] G. Frobenius & I. Schur – "Über die äquivalenz der Gruppen linearer Substitutionen.", *Berl. Ber.* **1906** (1906), p. 209–217 (German).

[4] F. Peter & H. Weyl – "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe", *Math. Ann.* **97** (1927), no. 1, p. 737–755.

[5] I. Schur – "Über eine Klasse von endlichen Gruppen linearer Substitutionen.", *Berl. Ber.* **1905** (1905), p. 77–91 (German).

[6] ———, "Neue Anwendungen der Integralrechnung auf Probleme der Ivariantentheorte. 1. Mitt.", *Berl. Ber.* **1924** (1924), p. 189–208 (German).

(4) Translator's note: Peter and Weyl use "unitäre Beschränkung" (unitary restriction), but I have found it reasonable to use the now-standard term.

(5) On this topic see [**7**], where the primitive characters of a semisimple group have been determined in explicit algebraic form.

[7] H. Weyl – “Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen. I, II, III, Nachtrag.”, *Math. Z.* **23** (1925), p. 271–309 (German).
[8] ———, “Integralgleichungen und fastperiodische Funktionen.”, *Math. Ann.* **97** (1926), p. 338–356 (German).

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