The tempered dual of real or p-adic groups and its noncommutative geometry

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joint with Anne-Marie Aubert

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$$\{ g \in GL(n, \mathbb{C}) : \quad \det(g) = 1 \}$$

$$\{ g \in GL(n, \mathbb{C}) : \quad g J g^t = J \} , \quad J \in \mathrm{Mat}(n, \mathbb{Z})$$

symmetric or antisymmetric

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Harish-Chandra, Langlands, ...

Representation theories of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ have fundamental features in common.

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- F : local field of characteristic zero
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- $G = \mathrm{G}(F)$, group of real points of a connected reductive group defined over F.

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- Examples : $\mathrm{GL}(n, F)$, $\mathrm{SL}(n, F)$, $\mathrm{Sp}(2n, F)$, $\mathrm{SO}(n, F)$...

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$$\mathbb{Z}_p = \left\{ \sum_{i=d}^{+\infty} a_i p^i \quad : \quad d \geqslant 0, \quad a_i \in \{0, \ldots, p-1\} \right\}$$

$$\begin{cases} (\mathbb{Q}_p)^\times / (\mathbb{Z}_p)^\times & \simeq \mathbb{Z} \\ (\mathbb{R}^\times)/\{\pm 1\} & \simeq \mathbb{R} \end{cases}$$

→ Characters of $GL(1, \mathbb{Q}_p)$ or $GL(1, \mathbb{R})^+$ that are trivial on every compact subgroup.

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- More generally, if F is a p-adic field with ring of integers $\mathfrak{O}_F$,

$$F^\times/(\mathfrak{O}_F)^\times \simeq \mathbb{Z}$$

Matrix elements and tempered representations

F local field of characteristic zero, $G = \mathrm{G}(F)$

- **Matrix elements**: for a unitary representation $(\pi, \mathcal{H})$ of G, and for $v, w \in \mathcal{H}$, define

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Map $G \longrightarrow \mathbb{C}$.

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Langlands (1970)	study of all "smooth" reps can be "reduced" to tempered case
Harish-Chandra + ...	study all tempered reps by induction from discrete series of certain subgroups of G.

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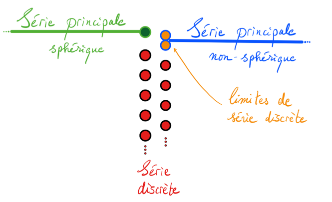
- **Tempered dual** $\mathrm{Irr}_{\mathrm{temp}}(G)$: set of equivalence classes of irreducible tempered reps. of G

$G = SL(2, \mathbb{R})$



- **Fell topology** on $\mathrm{Irr}_{\mathrm{temp}}(G)$ (compact convergence of matrix coefficients).

The tempered dual and its noncommutative geometry

- The tempered dual $\mathrm{Irr}_{\mathrm{temp}}(G)$ comes with a canonical non-Hausdorff topology.

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  The norm: for $f \in \mathcal{C}_c(G)$, set $\|f\| = \|\varphi \mapsto f \star \varphi\|_{\mathfrak{B}(\mathbf{L^2}(G))}$.

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- A landmark success: the Connes-Kasparov conjecture for the K-theory of $C^*_r(G)$.

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① $x \longrightarrow x_\infty$ in X

② $\Gamma_x \subset \Gamma_\infty$

③ $(\rho_\infty)|_{\Gamma_x}$ contains ρ_x

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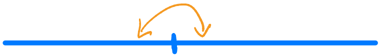

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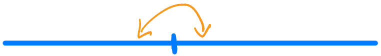

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- **A natural occurence:**

$(X/\!/\Gamma)_{\mathrm{spec}}$ = spectrum of the crossed-product C^*-algebra $\mathcal{C}_0(X) \rtimes \Gamma$

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Need to explain this now.

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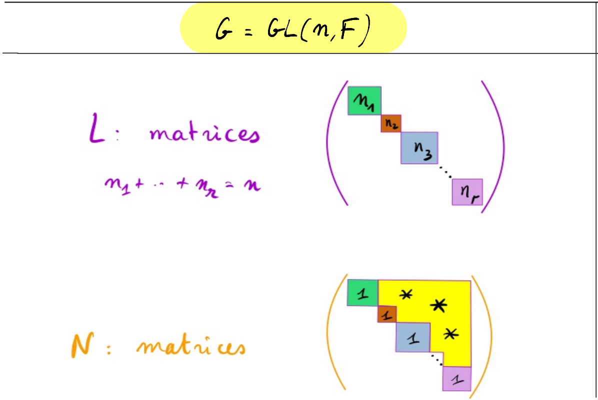

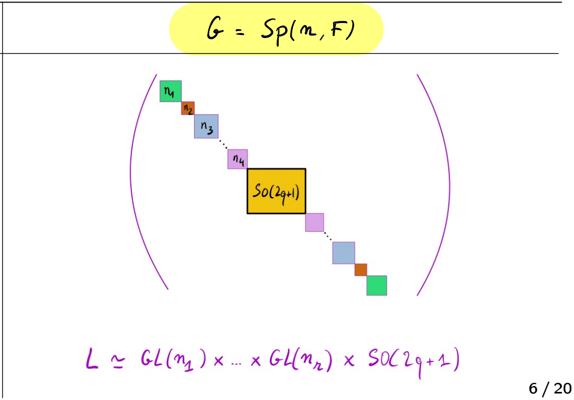

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in other words, trivial on $M = \langle$compact subgroups of $L\rangle$.

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Hilbert space: independent of χ

$$\mathcal{H} = \left\{ f \in L^2(K, V_\sigma) \text{ such that } f(k\ell n) = \sigma(\ell)^{-1} f(k) \text{ for } \begin{matrix} \ell \in L \cap K \\ n \in N \cap K \end{matrix} \right\}$$

where K is a maximal compact subgroup of G satisfying $G = KLN$.

G-action: depends on χ.

$\pi(g)$ sends f to $k \mapsto \chi(\lambda(g^{-1}k))\, \delta(\lambda(g^{-1}k))^{-1/2} f(\kappa(g^{-1}k))$

where $g^{-1}k = \underbrace{\kappa(g^{-1}k)}_{\in K} \underbrace{\lambda(g^{-1}k)}_{\in L} \underbrace{\nu(g^{-1}k)}_{\in N}$ and where $\delta : L \to \mathbb{R}^+$ is the modular fct of LN.

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G-action: depends on χ.

$\pi(g)$ sends f to $k \mapsto \chi(\lambda(g^{-1}k))\, \delta(\lambda(g^{-1}k))^{-1/2} f(\kappa(g^{-1}k))$

where $g^{-1}k = \underbrace{\kappa(g^{-1}k)}_{\in K} \underbrace{\lambda(g^{-1}k)}_{\in L} \underbrace{\nu(g^{-1}k)}_{\in N}$ and where $\delta : L \to \mathbb{R}^+$ is the modular fct of LN.

Can be reducible, but finite # of irred. components

Parabolic induction and connected components

Component of $\mathrm{Irr}_{\mathrm{temp}}(G) \leftrightarrow$ discrete pair $\Theta = (L, \sigma)$

- $P = LN$: parabolic subgroup of G
- σ: discrete series representation of L

- Form all induced representations $\mathrm{Ind}_{LN}^{G}(\sigma \otimes \chi)$, $\chi \in \mathcal{X}_u(L)$, finite number of irreducible factors
- Consider all of their irreducible factors

$\rightsquigarrow$ you get a subset $\widehat{G}_\Theta \subset \mathrm{Irr}_{\mathrm{temp}}(G)$.

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Théorème (Harish-Chandra + ...) :

The connected components of $\mathrm{Irr}_{\mathrm{temp}}(G)$ are exactly the sets $\widehat{G}_{\Theta}$.

- (L, σ) and (L, σ') yield the same component $\iff$ $\sigma' = \sigma \otimes \chi$ for some $\chi \in \mathcal{X}_u(L)$.
- Given L , component Θ $\leftrightarrow$ orbit $\mathcal{O}$ of $\mathcal{X}_u(L)$ inside $\mathrm{Irr}_{\text{discrete series}}(L)$.

First use of parabolic induction

Fix a connected component Θ corresponding to

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Want to define a C^*-algebra

$$\mathcal{C}_\Theta \subset C^*_r(G)$$

to study the connected component

$$\Theta \subset \mathrm{Irr}_{temp}(G)$$

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$$\left(\pi(f) = \int_G \pi(g) f(g) \, dg \right)$$

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$$f \longmapsto \left[\chi \mapsto \pi_\chi(f)\right]$$

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Certain finite groups can do this...

Weyl groups and intertwining operators

Fix a component Θ, and corresponding data:

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$$W_\Theta = \{ w \in W_L \ : \ w^\star\sigma \simeq \sigma \otimes \chi \text{ for some } \chi \in S \ \}$$

here $w^\star\sigma$ is

$$\begin{aligned} L &\longrightarrow \mathrm{End}(V_\sigma) \\ \ell &\longmapsto \sigma(w^{-1}\ell w) \end{aligned}$$

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not so easy ...

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Coming back to the C^*-morphism

$$C_r^*(G) \to \mathcal{C}_0(S, \mathfrak{K}(\mathcal{H})).$$

consider, on the right-hand side, the subalgebra $\mathcal{C}_\Theta \stackrel{\text{def}}{=} \mathcal{C}_0(S, \mathfrak{K}(\mathcal{H}))^{W_\Theta}$:

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Theorem (Plymen 1990) :

The map $C_r^*(G) \to \bigoplus_\Theta \mathcal{C}_\Theta$ is an isomorphism of C^*-algebras.

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That's our C^*-algebra attached to the component Θ.

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$$1 \to W'_{\sigma\otimes\chi} \to W_{\sigma\otimes\chi} \to R_{\sigma\otimes\chi} \to 1$$

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Theorem (Knapp-Stein for $G_{\mathbb{R}}$ – Silberger for G_F – around 1980) :

The irreducible components of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ are in natural bijection with $\mathrm{Irr}(R_{\sigma\otimes\chi})$.

Wassermann's theorem (1987)

In this slide $G = \mathrm{G}(\mathbb{R})$ is a real group

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- **Langlands decomposition:**

$$L = MA, \text{ direct product, and } \mathcal{X}_u(L) \simeq \widehat{A}.$$

- **A distinguished basepoint on the orbit $\mathcal{O}$:**
Twisting σ by $\sigma_{|A}^{-1} \in \mathcal{X}_u(L)$, we can assume $\sigma_{|A}$ trivial. Then

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Theorem (Wassermann 1987) :

The algebra $\mathcal{C}_\Theta$ is Morita-equivalent with $\mathcal{C}_0(S/W'_\sigma) \rtimes R_\sigma$

Corollary :

The connected component $\widehat{G}_\Theta$ is homeomorphic with $(S/W'_\sigma) /\!/ R_\sigma$

What about p-adic groups ?

Can we guess a p-adic theorem?

Fix a component Θ and corresponding data:

- L: Levi subgroup of G,
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- $S = \mathcal{X}_u(L)$ – **which is now a compact torus**.

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Irreducibility of all induced representations. Main example is $GL(n,F)$.

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- **Opdam & Solleveld, 2013 (extremely rough idea):**

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- **Opdam & Solleveld, 2013 (extremely rough idea):**
  A "small" open subset of $\widehat{G}_\Theta$ can be written as $(U/W_\sigma) /\!/ R_\sigma$, for $U \subset S$ "small".

A “unified” result

Properties of intertwining operaotrs

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$$\mathcal{A}(w_1 w_2, \sigma \otimes \chi) = \eta_\sigma(w_1, w_2)\mathcal{A}(w_1, w_2(\sigma \otimes \chi))\mathcal{A}(w_2, \sigma \otimes \chi)$$

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Key ingredients for both Wassermann and Plymen:

1. Compare $\begin{cases} [\mathcal{C}(S/W'_\sigma)\otimes\mathfrak{K}(\mathcal{H})]^{R_\sigma} & \text{(our } \mathcal{C}_\theta) \\ [\mathcal{C}(S/W'_\sigma)\otimes\mathrm{End}(\mathbb{C}[R_\sigma])]^{R_\sigma} & \text{(the crossed product } \mathcal{C}_0(S/W'_\sigma)\rtimes R_\sigma) \end{cases}$

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2. If $\eta_\sigma = 1$, then $w \mapsto \mathcal{A}(w, \bullet)$ is a 1-cocycle of W_σ with values in $\mathcal{C}(S, \mathfrak{K}(\mathcal{H}))^{W_\Theta}$

Arthur's central extension

$$\mathcal{A}(w_1 w_2, \sigma \otimes \chi) = \eta_\sigma(w_1, w_2)\mathcal{A}(w_1, w_2(\sigma \otimes \chi))\mathcal{A}(w_2, \sigma \otimes \chi)$$

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- The projective rep $r \mapsto \mathcal{A}(r, \sigma)$ becomes an authentic rep. $\tilde{r} \mapsto \mathcal{A}(\tilde{r}, \sigma)$ of $\tilde{R}_\sigma$.

Theorem (Arthur 1993) :

The representation $\tilde{R}_{\sigma\otimes\chi}$ on $\mathcal{H}$ is then **quasi-equivalent** with $\mathrm{Ind}_{Z_\sigma}^{\tilde{R}_\sigma}(\zeta)$.

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Can now compare $\begin{cases} (\mathcal{C}(S/W'_\sigma) \otimes \mathfrak{K}(\mathcal{H}))^{R_\sigma} & \text{(our } \mathcal{C}_\theta\text{)} \\ (\mathcal{C}(S/W'_\sigma) \otimes \mathrm{End}(\mathbb{C}[R_\sigma, \zeta]))^{R_\sigma} & \text{('twisted' crossed product)} \end{cases}$

What one can hope for in general

Fix a connected component Θ and

- L, σ: Levi subgroup and discrete rep. of L attached to Θ,
- $W_\sigma = W'_\sigma \rtimes R_\sigma$: Knapp-Stein decomposition for the stabilizer of σ in W_Θ
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No! Right-hand side is "too big":

remembers all reps. of $\tilde{R}_\sigma$,

not just those that occur in $\mathrm{Ind}_{Z_\sigma}^{\tilde{R}_\sigma}(\xi)$.

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Theorem (A. & A. – 2020) :

Assume that the following two conditions are satisfied:

1. $W_\sigma = W_\Theta$,
2. for all $\chi \in S$, there is an embedding $R_{\sigma\otimes\chi} \hookrightarrow R_\sigma$, and we have $W'_{\sigma\otimes\chi} \subset W'_\sigma$.

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- $F = \mathbb{R}$: recovers Wassermann's theorem (and fills in the details...)
- F p-adic: extends the results of Plymen & al.

What about the two hypotheses?

The case of classical p-adic groups

The groups and the simplifications

Quasi-split classical groups:

- $\mathrm{SL}(n, F)$
- split $\mathrm{SO}(2n+1, F)$
- $\mathrm{Sp}(2n, F)$
- quasi-split $\mathrm{SO}(2n, F)$
- $\mathrm{U}(n, n)$, $\mathrm{U}(n, n+1)$.

Simplifications:

- Levi subgroups and their Weyl groups are easily described

  ... 'block-diagonal matrices'...

- R-groups are of the form $(\mathbb{Z}/2\mathbb{Z})^r$, and much is known about them

Status of our two hypotheses?

- F : p-adic field
- G : quasi-split symplectic, orthogonal or unitary group over F

Start with a connected component Θ, and attach to it:

- a Levi subgroup L,
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Theorem (A. & A. – 2020) :

The action of W_Θ on $\mathcal{O}$ always admits a fixed point.

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... Definitely not ! We have classified the situations where this happens.

Conclusion

- The structure of $C_r^*(G)$ encodes many of the subtle phenomena in tempered representation theory,
- Some blocks in $C_r^*(G)$ have a particularly simple structure...
- ... understanding the structure of other blocks is wide open.

Thank you!

Existence of fixed points: an example

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  - $\mathsf{c}_2 \in W_\Theta \iff \sigma_2({}^t\bullet^{-1}) \simeq \sigma_2 \otimes \nu_2$ for some $\nu_2 \in \mathcal{X}_u(\mathrm{GL}(2, F))$